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Conical structures on the surface of a liquid with surface ionic conductivity: the space charge effect

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Abstract. Conical structures arising on the surface of a liquid with surface conductivity in an electric field are considered taking into account the influence of space charge of particles emanating from the cone apex. It is demonstrated that the field distribution problem admits an exact self-similar solution corresponding to the Taylor scaling law. This solution allows us to determine the relation between the total electric current flowing through the cone and the cone and space-charge region half-angles. The structure of this relation is quite sophisticated; it changes essentially with changing the permittivity of a liquid and the mobility of carriers.
Keywords: Taylor cone, liquid with surface conductivity, electric field, space charge.

1. Introduction

It is known that the free surface of liquids is unstable in a sufficiently strong external electric field. As a result, stationary conical structures can be formed on the surface. For the case of a perfectly conducting liquid, the conical formations were described in the pioneering work [1]. It was established that the cone half-angle is equal to the so-called Taylor angle, 49.3°. The electric field potential follows the scaling law $\Phi \sim R^{1/2}$ near the apex, where R is the distance from the singular point. This result was generalized to the case of an ideal dielectric liquid in [2]. The influence of the space charge of the droplets emanating from the apex of the cone on the surface of a conducting liquid was considered in [3]. It is remarkable that the electric field potential in the spray region also obeys Taylor's $R^{1/2}$ law. Recently, it was shown in [4, 5] that this scaling also remains valid if the surface ion current is taken into account.

In the present work, combining approaches developed in [3–5], we consider the conical formation on the surface of a liquid with surface (ionic) conductivity in an external electric field. Carriers move over the cone surface towards the apex, and charged particles (ions or droplets) emanating from the cone apex drift through the surrounding gas medium. The exact solution for the electric field distribution near the cone apex is found taking into account the influence of both the space and surface charges. The obtained solution makes it possible to determine the shape of the liquid cone depending on the value of electric current flowing over the cone surface. The results may be useful both for describing the operation of liquid sources of charged particles (ions or droplets) and for developing analytical methods for description of processes in a liquid phase at the electrode surface under conditions typical for the CLIC accelerating structures [6, 7].

2. Formulation of the problem

We assume that, under the action of the electric field, liquid takes the form of a cone with the apex angle 2β. The apex of the cone emits a flow of charged particles (ions or charged droplets for different applications) into the conic region of angle 2γ (see Fig. 1). These droplets drift outside the liquid cone in the surrounding gas medium.

Let us write down basic equations describing the electric field distribution around the cone taking into account the influence of space and surface charges. We will use the spherical coordinate system, where R is the distance from the cone apex and θ is the polar angle. Within the region $0 < \theta < \gamma$ (region I in Fig. 1), the space charge of drifting particles of density ρ is present. There is no space charge in the region $\gamma < \theta < \pi-\beta$ (region II) and also in the region $\pi-\beta < \theta < \pi$ (region III). There is a surface charge of density σ on the liquid cone surface $\theta = \pi-\beta$.

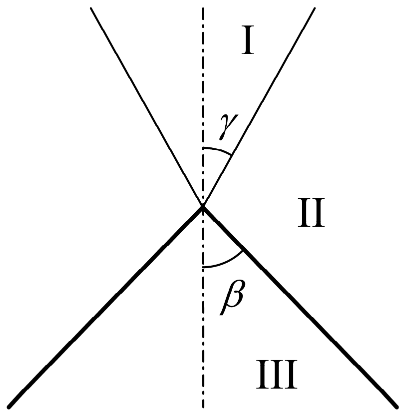

Fig. 1. Geometry of the problem.

The electric field potential Φ_1 in the region I obeys the Poisson equation, and the electric field potentials $\Phi_{2,3}$ in the regions II and III obey the Laplace equations. They should be solved together with necessary boundary conditions for components of the electric field on the boundaries of the regions I, II, and III. So, in particular, at $\theta = \pi-\beta$, it should be

$$\frac{\partial\Phi_2}{\partial R} = \frac{\partial\Phi_3}{\partial R}, \qquad \frac{1}{R}\left(\frac{\partial\Phi_2}{\partial\theta} - \varepsilon\frac{\partial\Phi_3}{\partial\theta}\right) = \frac{\sigma}{\varepsilon_0}, \tag{1}$$

where ε_0 is the electric constant and ε is the dielectric constant of liquid.

Let μ_1 and μ_2 be the mobilities of charged particles that move through the gas and, respectively, over the liquid surface. They drift with radial velocities given by

$$V_{1,2} = -\mu_{1,2}\frac{\partial\Phi_{1,2}}{\partial R}. \tag{2}$$

These velocities together with densities ρ and σ must satisfy the stationary continuity equations and also the condition of equality of the electric currents flowing over the surface of the liquid cone and through the surrounding gas.

Finally, the condition of equilibrium for the free liquid surface is the balance between the capillary and electrostatic pressures,

$$\frac{\alpha\cot\beta}{R} = \frac{\varepsilon_0}{2}\left(\frac{1}{R^2}\left(\frac{\partial\Phi_2}{\partial\theta}\right)^2 - \frac{\varepsilon}{R^2}\left(\frac{\partial\Phi_3}{\partial\theta}\right)^2 + (\varepsilon-1)\left(\frac{\partial\Phi_2}{\partial R}\right)^2\right), \tag{3}$$

where α is the surface tension coefficient of the liquid. This equation allows one to determine the cone angle.

3. Self-similar solution

Let us seek the solutions for the electric field potentials that satisfy the Taylor scaling law $\Phi_{1,2,3} \sim R^{1/2}$, which makes it possible to satisfy the pressure balance condition (3). As applied to the charge densities ρ and σ, this scaling law corresponds to the following relations: $\rho \sim R^{-3/2}$ and $\sigma \sim R^{-1/2}$. After simple algebra, we get $\Phi_1 = A{\cdot}R^{1/2}$, $\rho = B{\cdot}R^{-3/2}$, $\sigma = D{\cdot}R^{-1/2}$, and

$$\Phi_2 = C_1 P_{1/2}(\cos\theta) R^{1/2} + C_2 P_{1/2}(-\cos\theta) R^{1/2}, \qquad \Phi_3 = C_3 P_{1/2}(-\cos\theta) R^{1/2}, \tag{4}$$

where $P_{1/2}$ is the Legendre function of order 1/2 and A, B, $C_{1,2,3}$, D are some interrelated constants, which are determined by the problem parameters. As a result, we obtain the relation between the angles β and γ and the parameters $\mu_{1,2}$ and ε:

$$\frac{\left(1-\varepsilon - G(\gamma)G(\beta) - \varepsilon F(\beta)G(\gamma)\right)\left(F(\beta)G(\gamma)+1\right)}{\left(G(\gamma)+F(\gamma)\right)^2} = \frac{3m\left(1-\cos\gamma\right)P_{1/2}^2\left(\cos\gamma\right)}{4\sin^2\beta\, P_{1/2}\left(\cos\beta\right)P_{1/2}'\left(\cos\beta\right)}, \tag{5}$$

where m the mobility ratio ($m = \mu_1/\mu_2$) and we have denoted

$$F(x) = \frac{P_{1/2}\left(-\cos x\right)}{P_{1/2}\left(\cos x\right)}, \qquad G(x) = \frac{P_{1/2}'\left(-\cos x\right)}{P_{1/2}'\left(\cos x\right)}. \tag{6}$$

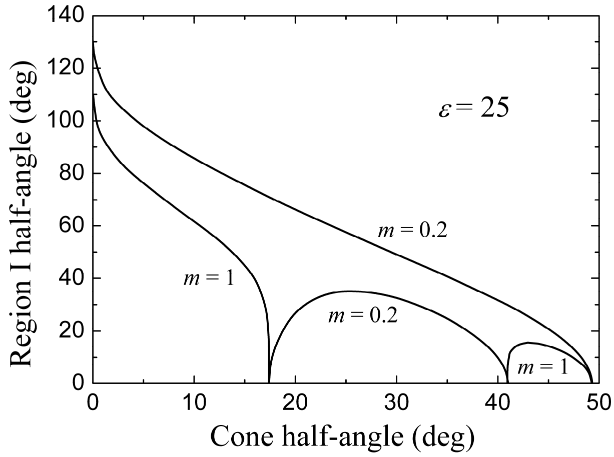

Fig. 2. Relations between the angles β and γ for ε = 25 and m = 0.2, 1.0.

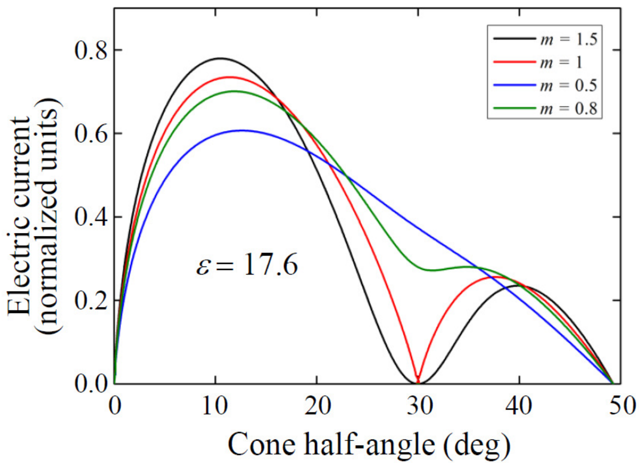

Fig. 3. Angular dependences of the current passing through the cone for ε = 17.6 and m = 0.5, 0.8, 1.0, 1.5.

As it turns out, the modes described by equation (5) are quite diverse. For example, the relation between the liquid cone half-angle β and the space charge region half-angle γ for ε = 25 and m = 0.2, 1.0 has a complicated form: see Fig. 2. The solutions have different branches; their structure changes significantly with changing the mobility ratio m. In any case, the possible half-angles of the stationary conical formations belong to the interval $0 < \beta < 49.3^o$, i.e., they cannot exceed the Taylor cone angle.

The electric current flowing through the conical structure is given by the expression

$$I = -\pi AB\mu_1 (1-\cos\gamma)\,. \qquad (7)$$

As an example, Fig. 3 shows the relation between I (it is normalized by its characteristic value $\alpha\mu_2$) and β for ε = 17.6 and m = 0.5, 0.8, 1.0, 1.5. One can see that it has complicated, nonmonotonic shape. Note that, at m = 0.5, the cusp arises in angular dependence of the current close to $\beta = 30^o$.

4. Conclusion

In the present work, the exact self-similar solution for the electric field distribution near the apex of the liquid cone has been found accounting the influence of both the surface charge of ions drifting towards the cone apex and the space charge of droplets emanating from the apex. The obtained solution allows us to determine the relation between the total electric current (I), the cone (β) and space-charge region (γ) half-angles, and such parameters of the system as the permittivity of the liquid (ε) and mobility of carriers ($\mu_{1,2}$). The solutions include, as particular cases, the solution from [3] (m = 0) and the solutions from [2] (γ = 0). It can be seen from Fig. 2 and Fig. 3 that our solutions have a rather complicated form and consist of different branches. As it turns out, their structure changes essentially with changing the mobility ratio and the permittivity.

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